Isotopic Spin of Exchanged Systems*

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It is demonstrated that any scattering amplitude can formally be expressed as a sum of terms each corresponding to the exchange of a particular isotopic spin without requiring the concept of crossing. Employing only the optical theorem and invariance under isotopic spin rotations the following theorem is proved: The contribution to the imaginary part of the forward scattering amplitude arising from the exchange of zero isotopic spin cannot be arbitrarily small compared to contributions from the exchange of any other isotopic spin or spins. From the same argument follows a second theorem: If the total cross section for two particles is independent of their isotopic spin state, then only the exchange of isotopic spin zero contributes to the imaginary part of the forward scattering amplitude. In combination with the hypothesis that high-energy scattering is dominated by the exchange of a single Regge pole, it follows that the "Pomeranchuk" pole must have isotopic spin zero. The relation to the Pomeranchuk-Okun' rule is discussed.

I. INTRODUCTION

IN 1956, Pomeranchuk and Okun'[1] suggested that at very high energies, forward-exchange amplitudes, and in particular charge-exchange amplitudes, are negligible compared with the forward-nonexchange amplitude. This is equivalent to the statement that the elastic-scattering amplitude matrix is diagonal in a representation in which the z projections of the isotopic spins of the individual particles are diagonal. Yang[2] has recently shown that this implies that the forward-scattering amplitude matrix is a multiple of the unit matrix and that this further implies that the total cross section is the same in all isotopic spin states. Various arguments may be given to support the Pomeranchuk-Okun' hypothesis, but none, to our knowledge, are of a rigorous character. On the other hand, the hypothesis has received increasing experimental support in recent years.

The past few years have also witnessed the appearance of the conjecture that the high-energy behavior of cross sections is dominated by the contributions of so-called Regge poles,[3] and this hypothesis has also been receiving experimental support. In the language of Regge poles the Pomeranchuk-Okun' rule can be expressed very simply: Forward-scattering amplitudes at sufficiently high energy are dominated by the exchange of a single Regge pole, the so-called "Pomeranchuk pole," which is characterized by certain quantum numbers, in particular isotopic spin zero. The dominance of this pole contribution thus ensures the dominance of charge-nonexchange over charge-exchange amplitudes. In the present note we establish a theorem on the basis of generally accepted premises which is closely related to, but weaker than, the Pomeranchuk-Okun' rule. The statement of the *theorem* is: If the imaginary part of the forward-scattering amplitude for two particles is analyzed into contributions from the exchange of various isotopic spins, then the contribution arising from the exchange of zero isotopic spin cannot be negligible compared with contributions from the exchange of any other isotopic spin or spins. In particular, if only one isotopic spin exchange contributes, it must be isotopic spin zero.

The basic premise underlying the theorem is invariance of the interactions under isotopic spin rotations, which limits its validity to those situations where strong interactions at least dominate the scattering process. Use is also made of the optical theorem to the extent of employing the fact that the forward-scattering amplitude in any isotopic spin state must have a positive imaginary part. In order for the theorem to have any content, it is, of course, necessary that there exist circumstances in which the description of an amplitude in terms of exchange of systems[4] of definite isotopic spin has some significance. Such circumstances occur in the conjecture that exchanges of Regge poles dominate high-energy cross sections. This conjecture, however, is based on the assumption of crossing relations between different channels and hence requires the possibility of analytically continuing amplitudes as functions of complex energy and momentum transfer through unphysical regions. For the purpose of establishing our theorem, no

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[1] I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **30**, 423 (1956) [translation: Soviet Phys.—JETP **3**, 306 (1956)]; L. B. Okun' and I. Ia. Pomeranchuk, *ibid.* **30**, 424 (1956) [translation: *ibid.* **3**, 307 (1956)].

[2] C. N. Yang, J. Math. Phys. **4**, 52 (1963).

[3] T. Regge, Nuovo Cimento **14**, 951 (1959); *ibid.* **18**, 947 (1960); R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962); G. F. Chew, S. C. Frautschi, and S. Mandelstam, *ibid.* **126**, 1202 (1962); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, *ibid.* **126**, 2204 (1962).

[4] To minimize ambiguities we reserve the term "particles" for the real entities undergoing scattering and the term "systems" for the entities exchanged between these. However, either or both of these may be "simple particles" or "complex systems."

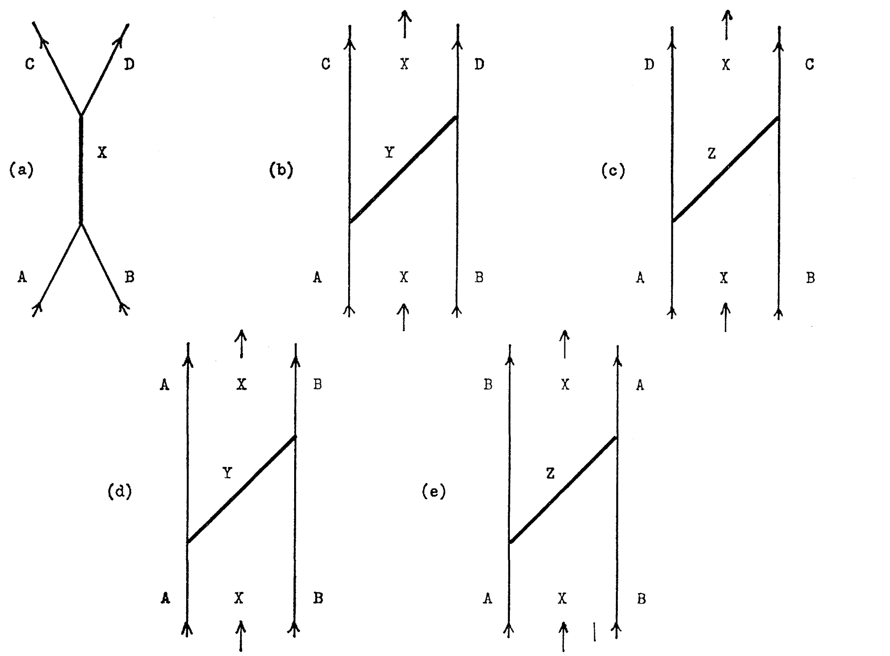

FIG. 1. Description schemes for scattering amplitudes. (a) Compound system scheme; (b) exchange system scheme; (c) "crossed" exchange scheme; (d) elast c scattering, exchange scheme; (e) elastic scattering, "crossed" exchange scheme.

such crossing relations are in fact needed, so we have avoided phrasing it in the language of Regge poles since it may have a wider applicability.

The arguments which establish the above theorem also serve to establish a *corollary*, namely: If the total cross section for two particles is the same in all isotopic spin states of the two particles, then only the exchange of isotopic spin zero is contributing to the imaginary part of the forward-scattering amplitude.

In Sec. II we give a formal definition of the amplitude for isotopic spin exchange. In Sec. III we prove the theorem and discuss possible generalizations, and in Sec. IV we discuss its consequences in terms of Regge poles.

II. EXCHANGE DESCRIPTION OF INTERACTIONS

Before specializing the discussion to the case of isotopic spin in particular, let us consider the concept of exchange in general in the description of a scattering reaction. Consider some reaction

$$A+B\rightarrow C+D \tag{1}$$

where the symbols A, B, C, D represent all the quantum numbers of the individual particles—energy-momenta, spin projections, charges, etc. We can define the amplitude $\langle CD|f|AB\rangle$, and this representation corresponds most closely to an actual measurement. For most discussions, however, it is convenient to transform to the representation labeled by the (total) conserved quantum numbers X of the system $A+B$, and write

$$\langle CD|f|AB\rangle=\sum_X C_X(ABCD)f_X, \tag{2}$$

where $C_X(ABCD)$ are some "kinematic" coefficients, i.e., independent of the exact dynamics of the interacting particles. In a certain sense this transformation may be regarded as expressing the amplitude as a sum of terms, each of which corresponds to an intermediate compound system with quantum numbers X, so that the reaction occurs in two stages [Fig. 1(a)]:

$$A+B\rightarrow X;\quad X\rightarrow C+D. \tag{3}$$

The concept of an intermediate compound state is particularly useful in the case of reactions proceeding through definite resonances, but the expansion (2) is a perfectly general one. In some cases, however, notably for stripping reactions, peripheral collisions, or the Regge pole theory of diffraction scattering, it is more convenient to regard the reaction as taking place via the exchange of some system[4] Y, the reaction again occurring in two stages [Fig. 1(b)]:

$$A\rightarrow C+Y;\quad Y+B\rightarrow D. \tag{4}$$

The system Y is rather an artificial concept; for example, its rest mass is imaginary. However, again one can consider all possible sets of quantum numbers Y consistent with (4) and work out the contributions $K_Y(ABCD)$ to various states $ABCD$ and write

$$\langle CD|f|AB\rangle=\sum_Y K_Y(ABCD)F_Y, \tag{5}$$

as an alternate formal expansion of the amplitude f. A third alternative is provided by the possibility of exchange in a "crossed" sense [Fig. 1(c)]:

$$A\rightarrow D+Z;\quad Z+B\rightarrow C, \tag{6}$$

for which the same comments apply. If we deal with an elastic scattering process with, say $A=C$, $B=D$, then Y exchange is generally the simpler description.

In the present note we are mainly concerned with isotopic spin. Let us consider a scattering process as discussed above proceeding via the exchange of a system Y. The amplitude is considered to correspond to some definite energy, momentum transfer, and spin orientations. A, B, C, D, and Y are now the (integral or half-integral) isotopic spins of the particles and the exchanged system.

As can be seen from the diagram [Fig. 1(b)] the reaction (4) involves an intermediate state with three isotopic spins: C, B, and Y.[5] The initial state corresponds to the coupling of C and Y in a state of isotopic spin A; the final state corresponds to the coupling of B and Y in a state of isotopic spin D. Thus, the transition from the initial to final state involves a transition from an eigenstate of one coupling scheme to an eigenstate of another for the three isotopic spins. The overlap between these two eigenstates, in so far as isotopic spin is concerned, depends only on the total isotopic spin X for which the reaction occurs. Thus, the relative contributions of the exchange of isotopic spin Y to the scattering of different isotopic spins X are just given by the recoupling coefficients[6]

$$\langle (CY)A,BX | C(YB)D,X\rangle = (-1)^{B+C+X+Y} \times[(2A+1)(2D+1)]^{1/2}\begin{Bmatrix} A & C & Y \\ D & B & X \end{Bmatrix}, \quad (7)$$

where $\begin{Bmatrix} A & C & Y \\ D & B & X \end{Bmatrix}$ is the 6-j symbol.[6]

In order to show that the transformation from the $ABCD$ representation to the Y representation is unitary, it is sufficient to show that there is a unitary transformation from the X to the Y representation. However, we must be careful about normalization: Any transformation must leave invariant the total transition probability summed over all charge states, and hence if f_X and F_Y are the amplitudes in the two representations,

$$\textstyle\sum_X (2X+1)|f_X|^2 = \sum_Y (2Y+1)|F_Y|^2. \quad (8)$$

In other words $(2X+1)^{1/2} f_X$ and $(2Y+1)^{1/2} F_Y$ must be related by a unitary transformation. This can be effected by writing

$$f_X = \textstyle\sum_X [(2Y+1)/(2X+1)]^{1/2} U_{XY}(ABCD) F_Y, \quad (9)$$

[5] Note that this exchange could also be considered in the reverse order: $B\rightarrow Y+D$, $A+Y\rightarrow C$. The definition of the exchange amplitude will, in fact, be invariant under the exchange $A\leftrightarrow B$, $C\leftrightarrow D$.

[6] See, for example, M. Rotenberg, R. Bivens, N. Metropolis, and J. K. Wooten, Jr., *The 3-j and 6-j Symbols* (Technology Press, Cambridge, Massachusetts, 1959), and references contained therein.

with

$$U_{XY} = U_{XY}(ABCD) = (-1)^{\frac{1}{2}(A+B+C+D+2X+2Y)} \times[(2X+1)(2Y+1)]^{1/2}\begin{Bmatrix} A & C & Y \\ D & B & X \end{Bmatrix}, \quad (10)$$

since the orthogonality relation for the 6-j symbols[7] can be expressed as

$$\textstyle\sum_X U_{XY'}^* U_{XY} = \delta_{Y'Y}. \quad (11)$$

There is some arbitrariness in the phase factor of U_{XY} reflecting the arbitrariness in the phases of the F_Y. Our particular choice is dictated by convenience and has the advantage that there is symmetry under the interchange: $A\leftrightarrow B$, $C\leftrightarrow D$, as well as under the interchange: $A\leftrightarrow C$, $B\leftrightarrow D$; furthermore the U_{XY} are real, for $A=C$, $B=D$. The use of the orthogonality relation (11) allows us to solve (9) for the F_Y

$$F_Y = \textstyle\sum_X [(2X+1)/(2Y+1)]^{1/2} U_{XY}^* f_X. \quad (12)$$

This equation can then be regarded as *defining*[8] the amplitude for *exchange* of isotopic spin Y. The term containing F_Y in (9) is then by definition the contribution to the amplitude f_X arising from the exchange of systems of isotopic spin Y.

III. THEOREM AND DISCUSSION

We begin the proof of our theorem by remarking that the Racah coefficient $W(Y,C,B,X;A,D)$ is defined in terms of the 6-j symbol by[9]

$$W(Y,C,B,X;A,D) = (-1)^{-(B+C+X+Y)}\begin{Bmatrix} Y & C & A \\ X & B & D \end{Bmatrix} = (-1)^{-(B+C+X+Y)}\begin{Bmatrix} A & C & Y \\ D & B & X \end{Bmatrix}, \quad (13)$$

where the last equality follows from the symmetry properties of the 6-j symbol. We now assume that $A>C$ and that $A-C>|B-D|$.[10] In this case $A-C$ is the smallest isotopic spin which can be exchanged between

[7] Reference 6, p. 14, Eq. (2.6).

[8] This result is equivalent to the expression of a crossing matrix in terms of a 6-j symbol or Racah coefficient but is independent of any crossing assumption. See, for example, reference 2. The first application of Racah coefficients to crossing matrices seems to have been made by F. J. Dyson, Phys. Rev. **100**, 344 (1955) in a context representative of the general situation considered here. See also G. C. Wick, Brookhaven Lectures, 1960 (unpublished).

[9] Reference 6, p. 13, Eq. (2.1).

[10] If this is not the case, we may use the symmetry properties of the 6-j symbol to rearrange the elements of the first two columns until the symbol has the form $\begin{Bmatrix} P & R & Y \\ S & Q & X \end{Bmatrix}$, with $P>R$ and $P-R>|Q-S|$. In this case $P-R$ is the smallest isotopic spin which can be exchanged. The argument which follows then goes through unchanged with the substitutions $A\rightarrow P$, $B\rightarrow Q$, $C\rightarrow R$, $D\rightarrow S$.

the particles. We next make the important remark that[11] the Racah coefficient $W(A-C,C,B,X;A,D)$ is *positive* for all values of its arguments for which it does not vanish; in particular, it is positive definite for $A-C\leqslant X\leqslant A+C$. Thus, setting $Y=A-C$ in (13) and comparing with Eq. (10), it follows that

$$U^*_{X,A-C}=(-1)^{-\frac{1}{2}(A-B-C+D)}[(2X+1)(2A-2C+1)]^{1/2} W(A-C,C,D,X;A,B) \quad (14)$$

whether real or imaginary, is of one sign for all pertinent X. In particular for $A=C$, $B=D$, one finds

$$U_{X,0}=U_{X,0}^*=[(2X+1)/(2A+1)(2B+1)]^{1/2}.$$

If we now set $Y'=A-C$ in the orthogonality relation (10), we obtain

$$\sum_X U_{X,A-C}^* U_{X,Y}=\delta_{Y,A-C}.$$

Multiplying by $(2Y+1)^{1/2}F_Y$ and summing over all Y *except* $A-C$, the right side vanishes and we find

$$\sum_X (2X+1)U^*_{X,A-C} \times\{\sum_{Y\neq A-C}[(2Y+1)/(2X+1)]^{1/2}U_{XY}F_Y\}=0. \quad (15)$$

Now since $(2X+1)U^*_{X,A-C}$ is of one algebraic sign and nonzero for all values of X included in the sum, it follows that neither the real nor the imaginary part of the quantity in braces can have the same algebraic sign for all X, no matter what the values of the F_Y.

The preceding statement is a general theorem about 6-j symbols, but in the context of our earlier discussion, with the quantities F_Y representing the contributions arising from the exchange of systems with isotopic spin Y, the quantity in braces in Eq. (15) will be recognized as f_X in the case that $F_{A-C}=0$. Let us now apply this result to forward elastic scattering described by the exchange scheme depicted in Fig. 1(d), so that $A=C$ and $B=D$. Then $A-C=0$ and the above theorem then states that if F_0 is negligible compared to the other F_Y, the imaginary part of f_X cannot be positive for all X. Since this would violate the optical theorem, it follows that the contribution to the imaginary part of the forward-scattering amplitude arising from the exchange of zero isotopic spin between the scattering particles cannot be negligible compared to contributions arising from the exchange of other isotopic spins. In particular, the only F_Y which can dominate all the others is F_0, and if this is the case, Eq. (9) then yields

$$f_X=F_0/[(2A+1)(2B+1)]^{1/2}. \quad (16)$$

The total cross section is then the same in all isotopic spin states X, in accord with Yang's conclusion from the Pomeranchuk-Okun' hypothesis. We see further that since the contributions to the imaginary part of f_X from the exchange of other isotopic spins than zero cannot be of the same sign for all X, we have also proved that if the total cross section for two particles is the same in all isotopic spin states then only the exchange of isotopic spin zero is contributing to the imaginary part of the forward-scattering amplitude.

As a particular example of the above results, consider the elastic scattering of two nucleons. In this case we have an explicit representation of the matrix U in the form

$$f=\tfrac{1}{2}F_0-\tfrac{1}{2}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2F_1, \quad (17)$$

where f is a matrix in the isotopic spin space of the two nucleons and $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are their isotopic spin vectors. Letting g and G represent the imaginary parts of f and F, respectively, we then have

$$g_0=\tfrac{1}{2}G_0+\tfrac{3}{2}G_1,$$
$$g_1=\tfrac{1}{2}G_0-\tfrac{1}{2}G_1, \quad (18)$$

and it is clear that we must also have the inequalities

$$G_0\geqslant 0,\quad G_0\geqslant G_1\geqslant -G_0/3. \quad (19)$$

In the general case similar inequalities may be derived.

It is clear from the above proof that similar theorems could be established for the real or imaginary part of any amplitude once one has independent arguments that these are of a definite sign in all isotopic spin states. The authors believe also that in the particular case of forward elastic scattering it is possible to define in an unambiguous way, and without reference to crossing relations, the exchange of spin, parity, and signature between the two scattering particles, and further, that with this definition one can show that the contribution from the exchange of zero spin, even parity, and even signature cannot be negligible and is the only contribution allowed to dominate. This has not yet been worked out in detail, however, and hence is postponed for a later communication. A further extension of considerations of the above character to other conserved quantum numbers associated with invariance under some other group (SU_3, for example) would require the generalizations of the 6-j symbols for this group, an appropriate orthogonality relation, etc.

Though of lesser interest, one can also consider elastic scattering but according to the alternate "crossed" exchange scheme depicted in Fig. 1(e). This corresponds to the identification $A=D$, $B=C$, $Y\rightarrow Z$, in Eq. (15). In this case $A-B$ is the minimum isotopic spin which can be exchanged between the two particles, and hence if a single isotopic spin exchange is to dominate the imaginary part of the forward-scattering amplitude in this exchange scheme, it must be the minimum exchangeable isotopic spin.

IV. APPLICATION TO THE REGGE POLE HYPOTHESIS

While we have expressed our theorems in a form which does not depend directly on crossing relations, the

[11] L. C. Biedenharn, J. M. Blatt, and M. E. Rose, Rev. Mod. Phys. 24, 249 (1952), Eq. (29). For the particular case $A=C$, $B=D$, see reference 6, p. 16, Eq. (2.12).

physical significance of a general expansion in terms of exchanged-quantum numbers is most likely to be relevant to situations involving crossing. It may, in fact, be the case that the applicability of the theorems is limited in practice to those situations where the Regge pole hypothesis is realized.

We, therefore, conclude with a brief discussion of the implications of our first theorem for the Regge pole hypothesis. According to this hypothesis,[3] at sufficiently high energies each of the contributions F_Y to the forward-scattering amplitude has a dependence on t, the square of the center-of-mass energy of the scattering particles, of the form

$$F_Y = iR_Y t^{\alpha_Y}.$$

Here α_Y is the position on the real axis in the complex angular momentum plane (for zero center-of-mass energy in the crossed channel) of the "dominant" Regge pole having total isotopic spin Y (and appropriate other good quantum numbers). Our theorem then implies

$$\alpha_Y \leqslant \alpha_0 \quad (\text{all } Y \neq 0).$$

If the inequality holds in all cases, then we have the dominance of the pole with isotopic spin zero in agreement with the Pomeranchuk hypothesis. If the equality holds for one or more Y (coincidence of Regge poles with different isotopic spins), then we can only conclude that some inequalities hold between the real parts of the coefficients R_Y. In the particular case of nucleon-nucleon scattering, for example, one would have

$$R_0 \geqslant R_1 \geqslant -R_0/3.$$

On the other hand, our second theorem allows us to infer from the observed equality of n-p and p-p total cross sections, as well as $\pi^+ - p$ and $\pi^- - p$ total cross sections at very high energies that only (one or more) isotopic spin zero Regge poles are contributing to the forward scattering in this energy range.

Pomeranchuk has further suggested (and present experiments are consistent with this suggestion) that total cross sections are asymptotically constant at very high energies, implying $\alpha_0 = 1$; our considerations shed no light on this point.

If indeed we are able to extend our theorem to the case of ordinary spin, parity, and signature,[3] then we will have shown that the "observed" quantum numbers —isotopic spin zero, even parity, and even signature— of the "Pomeranchuk" trajectory are the only ones allowed for a single dominant exchanged Regge pole.

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